**Two sums**

https://www.linkedin.com/feed/update/urn:li:activity:6537556324908171264

Evaluate

**(i)** $S = \sum_{k=1}^{n} \frac{k}{(2k-1)(2k+1)(2k+3)}$.

**(ii)** $S = \sum_{k=1}^{n} \frac{k^4}{(2k-1)(2k+1)}$.

**Solution by Arkady Alt, San Jose, California, USA.**

**(i)** Since $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) =$
$\frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{n}{2n+1}$ and $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)(2k+3)} =$
$\frac{1}{4}\sum_{k=1}^{n}\left(\frac{1}{(2k-1)(2k+1)} - \frac{1}{(2k+1)(2k+3)}\right) =$
$\frac{1}{4}\left(\frac{1}{3} - \frac{1}{(2n+1)(2n+3)}\right) = \frac{n(n+2)}{3(2n+3)(2n+1)}$ then
$2S = \sum_{k=1}^{n} \frac{2k+3-3}{(2k-1)(2k+1)(2k+3)} = \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} -$
$3\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)(2k+3)} = \frac{n}{2n+1} - \frac{n(n+2)}{(2n+3)(2n+1)} =$
$\frac{n(n+1)}{(2n+3)(2n+1)} \Leftrightarrow S = \frac{n(n+1)}{2(2n+3)(2n+1)}$.

**(ii)** Since $(4k^2-1)^2 = 16k^4 - 2(4k^2-1) - 1 \Leftrightarrow$
$16k^4 = (4k^2-1)^2 + 2(4k^2-1) + 1$ then $16S = \sum_{k=1}^{n} \frac{16k^4}{4k^2-1} =$
$\sum_{k=1}^{n}(4k^2-1) + \sum_{k=1}^{n} 2 + \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} =$
$4\sum_{k=1}^{n} k^2 + n + \frac{n}{2n+1} = 4 \cdot \frac{n(n+1)(2n+1)}{6} + n + \frac{n}{2n+1} =$
$\frac{8n(n+1)(n^2+n+1)}{3(2n+1)} \Leftrightarrow S = \frac{n(n+1)(n^2+n+1)}{6(2n+1)}$ :